

# Extreme flatness and Hahn-Banach type theorems for normed modules over $c_0$ <sup>1</sup>

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## Introduction: formulation of the main results and comments

In this paper we consider a certain specific case of a well known typical question in the theory of normed algebras and their modules. This is a question about conditions that ensure the preservation of isometries under projective tensor product of modules. Such a question is intimately connected with the problem of extension of a given bounded morphism from a submodule to a bigger module with the preservation of its norm. In other words, it is connected with the question of the existence, in certain situations, of module versions of the classical Hahn-Banach Theorem.

We proceed to relevant formal definitions.

Let  $A$  be a normed algebra. We shall use the symbol ' $\otimes_A$ ' for the *non-completed* projective module tensor product of  $A$ -modules and of their bounded morphisms. (See, e.g., [1] or, as to the initial 'completed' version, the pioneering paper of Rieffel [2] or the textbooks [3, II.3] [4, VI.3.2]).

The identity operator on a linear space (or a module)  $Z$  will be denoted by  $\mathbf{1}_Z$ , or just  $\mathbf{1}$ , if there is no danger of misunderstanding.

Let us distinguish a class, so far arbitrary, of right normed  $A$ -modules and denote it by  $\mathcal{K}$ . In the spirit of the well-known definitions of a flat and of a strictly flat Banach module ([3, VII.1], [4, VII.1.3]), we give the following

**Definition.** A normed left  $A$ -module  $Z$  is called *extremely flat with respect to the class  $\mathcal{K}$*  or, for short,  *$\mathcal{K}$ -E-flat*, if, for every isometric morphism  $i : X \rightarrow Y$  of right modules, belonging to  $\mathcal{K}$ , the operator  $i \otimes_A \mathbf{1}_Z : X \otimes_A Z \rightarrow Y \otimes_A Z$  is also isometric.

If  $A := \mathbb{C}$ , that is if we deal with just normed spaces, the well known theorem of Grothendieck [5, Thm. 1], being adapted to non-complete spaces, gives a full description of the extremely flat objects in the following way. *A normed space ('normed  $\mathbb{C}$ -module') is extremely flat with respect to the class of all normed spaces*

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if and only if it is isometrically isomorphic to a dense subspace of  $L_1(\Omega, \mu)$  for some measure space  $(\Omega, \mu)$ .

**Definition.** A normed right  $A$ -module  $Z$  is called *extremely injective with respect to the class  $\mathcal{K}$*  or, for short,  *$\mathcal{K}$ -E-injective*, if, for every isometric morphism  $i : X \rightarrow Y$  of right modules, belonging to  $\mathcal{K}$ , and for every bounded morphism  $\varphi : X \rightarrow Z$  of right  $A$ -modules, there exists a bounded morphism of right modules  $\psi : Y \rightarrow Z$  such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow \varphi & \nearrow \psi & \\ Z & & \end{array}$$

is commutative and  $\|\varphi\| = \|\psi\|$ . In other words, every bounded morphism of right modules from  $X$  into  $Z$  can be extended, after the identification of  $X$  with a submodule of  $Y$ , to a morphism from  $Y$  to  $Z$  with the same norm.

Thus the assertion that a certain module  $Z$  is  $\mathcal{K}$ -E-injective can be considered as a ‘Hahn-Banach type’ theorem for given  $A$  and  $\mathcal{K}$ , with  $Z$  playing the role of  $\mathbb{C}$  in the mentioned classicial theorem.

If again  $A := \mathbb{C}$ , then the extremely injective objects are described by a theorem, connected with the names of Nachbin, Goodner, Hasumi and Kelley (see [6, p.123] or [7, Thm. 25.5.1]), which can be easily adapted to the non-complete case. Namely, *a normed space is extremely injective with respect to the class of all normed spaces if and only if it is isometrically isomorphic to the space  $C(\Omega)$ , where  $\Omega$  is an extremely disconnected compact space.*

**Remark.** The word ‘extremely’ in both definitions is chosen because isometric operators or morphisms are exactly the so-called extreme monomorphisms in some principal categories of spaces or modules in functional analysis (cf., e.g., [8, p. 4], [9, Ch. 0.5]).

The both introduced notions are closely connected. The link is provided by a proper functional-analytic version of the algebraic ‘law of adjoint associativity’. This version was established by Rieffel [10] (who considered Banach modules). With its help, we shall prove below (see the beginning of Section 2) the following easy

**Proposition.** *Let  $A, \mathcal{K}$  and  $Z$  be as above. Then  $Z$  is  $\mathcal{K}$ -E-flat if and only if its dual normed left  $A$ -module  $Z^*$  is  $\mathcal{K}$ -E-injective.*

The notions, defined above, were actually introduced in [11], however, for only some special algebras and modules. Namely, the role of a base algebra was played by  $\mathcal{B}(H)$  for a Hilbert space  $H$ , and the class  $\mathcal{K}$  consisted of the so-called semi-Ruan  $\mathcal{B}(H)$ -modules. (Speaking informally, these are modules, satisfying a proper one-sided version of Ruan axioms for an operator space; cf. [1]). It was shown

that certain  $\mathcal{B}(H)$ -modules are extremely flat with respect to that  $\mathcal{K}$ , and certain Hahn-Banach type theorems for modules over  $\mathcal{B}(H)$  were obtained as corollaries. These theorems, in their turn, led to a transparent new proof of one of basic theorems of operator space theory, the Arveson-Wittstock Theorem about extensions of completely bounded operators (see, e.g., [12] or [1]).

Afterwards the results of [11] were generalized and considerably strengthened by Wittstock [13], who, in particular, replaced  $\mathcal{B}(H)$  by an arbitrary properly infinite  $C^*$ -algebra and established that every semi-Ruan module is  $\mathcal{K}$ -E-flat. As an application of his results, Wittstock presented a new transparent proof of the Arveson-Wittstock Theorem in a more sophisticated version, that for operator modules.

After the cited papers it seemed natural to look for extremely flat modules over other classes of normed algebras and, accordingly, for related Hahn-Banach type theorems. In particular, what can we find, if we turn to commutative algebras? This class, in a sense, is opposite to highly non-commutative algebras of [11] and [13].

In the present paper we exclusively deal with the apparently simplest of all infinite-dimensional commutative normed algebras. This is the algebra  $c_0$  of complex-valued sequences, converging to 0, with the coordinate-wise operations and the uniform norm. It turned out that even in this case there is something to say. (Speaking very roughly, extremely flat  $c_0$ -modules form much larger family that one could initially expect).

We recall that a normed module  $X$  over a normed algebra  $A$  is called *contractive*, if we have  $\|a \cdot x\| \leq \|a\|\|x\|$ , or, accordingly,  $\|x \cdot a\| \leq \|a\|\|x\|$  for all  $a \in A, x \in X$ . *Throughout this paper, all normed modules are always supposed to be contractive.*

If  $A$  and  $X$  are as before, we denote the closure of the linear span of the set  $\{a \cdot x : a \in A, x \in X\}$  by  $X_{es}$  and call it *essential part of  $X$* . It is, of course, a submodule. A left  $A$ -module  $X$  is called *essential* (they often say also ‘non-degenerate’), if we have  $X = X_{es}$ . The quotient normed  $A$ -module  $X/X_{es}$  is denoted by  $X_{an}$ ; obviously it has zero outer multiplication. The *annihilator of  $A$  in  $X$*  is the closed left submodule  $\{x : a \cdot x = 0 \text{ for all } a \in A\}$  in  $X$ , denoted by  $AnnX$ . The quotient left normed  $A$ -module  $X/AnnX$  is called the *reduced module of  $X$*  and denoted by  $X^{red}$ .

As usual, we call a left  $A$ -module  $X$  *faithful*, if  $AnnX = 0$ . Of course, the reduced module of every module is faithful. It is easy to show that every essential left  $A$ -module is faithful provided  $A$  has a bounded left approximate identity.

Recall what happens if  $A$  is commutative, as it is the case with  $c_0$ . Then every left  $A$ -module is a right  $A$ -module with the same bilinear operator of the outer multiplication, and vice versa. Therefore we identify both types of modules and say just ‘ $A$ -module’. Accordingly, we can speak about module projective tensor product of two normed  $A$ -modules and of two bounded morphisms of normed  $A$ -modules.

Moreover, we immediately see that the mentioned tensor product of two modules, say  $X$  and  $Y$ , is itself a normed contractive  $A$ -module with the outer multiplication, well defined by  $a \cdot (x \underset{A}{\otimes} y) := (a \cdot x) \underset{A}{\otimes} y$  (or  $:= x \underset{A}{\otimes} (a \cdot y)$ ). Besides, the mentioned tensor product of two bounded morphisms of normed  $A$ -modules is obviously itself a bounded morphism of the respective modules.

The main result of the paper gives, within a certain reasonable class of normed  $c_0$ -modules, a full description of extremely flat modules with respect to that class. After some preliminary note, we proceed to the definition this class.

One can immediately see, what makes the work with  $c_0$  easier than with other algebras. It is the presence in this space of a distinguished countable Schauder basis, consisting of irreducible idempotent generators. We mean, of course, the ‘orts’  $(0, \dots, 0, 1, 0, 0, \dots) \in c_0$ . The  $n$ -th ort (that with 1 as its  $n$ -th term) will be denoted by  $\mathbf{p}^n$ . If  $X$  is a normed  $c_0$ -module, we set  $X_n := \{\mathbf{p}^n \cdot x; x \in X\}$  for every  $n = 1, 2, \dots$ . We see that  $X_n$  is a submodule of  $X$ ; it will be called the  $n$ -th *coordinate submodule*. Often, when there is no danger of confusion, for  $x \in X$  we shall write  $x_n$  instead of  $\mathbf{p}^n \cdot x$ . Of course, we have  $\mathbf{p}^n \cdot x_n = x_n$ .

**Definition.** A  $c_0$ -module  $X$  is called *homogeneous* if, for every  $x, y \in X$ , the equalities  $\|x_n\| = \|y_n\|$ , for all  $n$ , imply that  $\|x\| = \|y\|$ .

In particular, all essential normed  $c_0$ -modules, consisting of complex-valued sequences, are homogeneous (Proposition 3.1 below). Besides,  $l_p$ -sums;  $1 \leq p \leq \infty$  of arbitrary families of normed spaces are obviously homogeneous. (In both cases we mean the coordinate-wise outer multiplication).

It is evident that every homogeneous normed  $c_0$ -module is faithful.

In this paper, by  $\mathcal{H}$  we denote the class of all homogeneous normed  $c_0$ -modules, and by  $\mathcal{H}_{es}$  its subclass, consisting of essential modules.

**Theorem I.** *Let  $Z$  be an essential (respectively, arbitrary) homogeneous normed  $c_0$ -module. Then  $Z$  is extremely flat with respect to  $\mathcal{H}$  (respectively, with respect to  $\mathcal{H}_{es}$ ) if and only if, for every  $n$ , its  $n$ -th coordinate submodule is isometrically isomorphic to a dense subspace of the space  $L_1(\Omega_n, \mu_n)$  for some measure space  $(\Omega_n, \mu_n)$ .*

Note that ‘only if’ part of this theorem relies heavily on the theorem of Grothendieck, cited above, and it is rather easy corollary of the latter. As to the ‘if’ part, our proof of this is more complicated, and it does not use the Grothendieck Theorem).

In fact, we shall prove this theorem in a slightly stronger form; see Proposition 3.3 and Theorem 3.7 below.

The following theorem is a rather easy corollary of Theorem I.

**Theorem II** (see end of Section 4). *Let  $Z$  be an essential (respectively, arbitrary) homogeneous normed  $c_0$ -module. Then the dual module  $Z^*$  is extremely injective with respect to  $\mathcal{H}$  (respectively, with respect to  $\mathcal{H}_{es}$ ) if and only if for every  $n$  we have that its  $n$ -th coordinate submodule  $(Z^*)_n$  is isometrically isomorphic to the Banach space  $L_\infty(\Omega_n, \mu_n)$  for some measure space  $(\Omega_n, \mu_n)$ .*

In particular, all  $c_0$ -modules  $l_p$ ;  $1 \leq p < \infty$  are  $\mathcal{H}$ -E-flat whereas the same  $l_p$  and also  $l_\infty$  are  $\mathcal{H}$ -E-injective.

In both theorems we assumed that some participating modules are essential. Such a condition can not be omitted: a non-essential homogeneous normed module (being always  $\mathcal{H}_{es}$ -E-flat) is not bound to be  $\mathcal{H}$ -E-flat. As a matter of fact, *the  $c_0$ -module  $l_\infty$  (apparently the first faithful non-essential  $c_0$ -module that comes in mind), is not extremely flat with respect to the class of all homogeneous modules.* This is Theorem 4.3.

Let us make some comments on the proof of the main result. In the very beginning we observe that, under some conditions, tensor products of  $c_0$ -modules and their morphisms can be described in a rather transparent and ‘workable’ form (Proposition 1.6). In particular, this is helpful in making the principal preparatory step, Lemma 3.4 of somewhat technical character. At the end of our argument, we have used the following fact: if  $X$  or  $Z$  are essential, then the property of  $\varphi : X \rightarrow Y$  to be (just) injective implies the same property of  $\varphi \otimes_{c_0} \mathbf{1}_Z$ .

Thus, trying to prove the preservation of isometries, we came across another typical question of the theory of normed algebras. Which conditions ensure the preservation, under projective tensor multiplication of modules, of the property of a given morphism to be injective? We believe that such a question deserves to be considered independently. Of course, it sounds similar to its well known pure algebraic prototype, which leads to the fundamental notion of the (algebraic) flatness. But here we deal with the bounded morphisms and a kind of functional-analytic tensor product. This profoundly affects the situation.

As a matter of fact (see Example 2.3), if  $X, Y, Z$  are normed  $c_0$ -modules, even consisting of sequences, then it can well be that a bounded morphism  $\varphi : X \rightarrow Y$  is injective whereas  $\varphi \otimes_{c_0} \mathbf{1} : X \otimes_{c_0} Z \rightarrow Y \otimes_{c_0} Z$  is not. However, if we are given arbitrary normed  $c_0$ -modules  $X, Y, Z$  and a *topologically injective* (in particular, isometric) morphism  $\varphi : X \rightarrow Y$  then *the operator  $\varphi \otimes_{c_0} \mathbf{1} : X \otimes_{c_0} Z \rightarrow Y \otimes_{c_0} Z$  is also injective.* (Note that at the same time it is not bound to be again topologically injective). This is the future Theorem 2.4.

**Remark.** We want to emphasize that we work in this paper, in a similar way as in [11][13], with the non-completed version of the module projective tensor product.

If we replace the latter by the respective completed version, Theorem 2.4 fails to be true. One can easily construct respective counter-examples, taking some spaces without the approximation property.

## 1. Some preparations

We begin our preliminaries with a proposition of somewhat general character. In particular, it will enable us to derive Theorem II from Theorem I. This proposition actually appeared in [11, Prop. 9], but in a certain special case and in a slightly disguised form.

In what follows  $A$  is a normed algebra, so far arbitrary, and  $\mathbf{h}_A(\cdot, \cdot)$  is the symbol of the space of all bounded morphisms between right normed modules. Such spaces are equipped with the operator norm.

**Proposition 1.1.** *Let  $X$  and  $Y$  be right normed  $A$ -modules,  $Z$  a left normed  $A$ -module,  $i : X \rightarrow Y$  an isometric morphism,  $Z^*$  the right Banach  $A$ -module, dual to  $Z$ . Then the following statements are equivalent:*

- (i) *the operator  $i \otimes \mathbf{1}_Z : X \otimes_A Z \rightarrow Y \otimes_A Z$  is an isometry*
- (ii) *for every bounded morphism  $\varphi : X \rightarrow Z^*$  of right  $A$ -modules, there exists a bounded morphism of right modules  $\psi : Y \rightarrow Z^*$  such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow \varphi & \swarrow \psi & \\ Z^* & & \end{array}$$

*is commutative and  $\|\varphi\| = \|\psi\|$ .*

◁ According to the functional-analytic version of the law of the adjoint associativity (cf. [10] or [1, Ch. 8.0]) the normed space  $\mathbf{h}_A(X, Z^*)$  coincides with the space  $(X \otimes_A Z)^*$  up to the isometric isomorphism, taking a morphism  $\varphi : X \rightarrow Z^*$  to the functional  $f : X \otimes_B Z \rightarrow \mathbb{C}$ , well-defined by  $f(x \otimes_A z) = [\varphi(x)](z)$ . Similarly,  $\mathbf{h}_A(Y, Z^*)$  is identified with  $(Y \otimes_A Z)^*$ . Moreover, one can easily check that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{h}_A(Y, Z^*) & \xrightarrow{i_*} & \mathbf{h}_A(X, Z^*) \\ \downarrow & & \downarrow \\ (X \otimes_A Z)^* & \xrightarrow{i^\bullet} & (Y \otimes_A Z)^* \end{array}$$

Here the vertical arrows depict isometric isomorphisms of normed spaces, acting as it was indicated,  $i_*$  acts as  $\beta \mapsto \beta i$ , and  $i^\bullet$  is the operator which is adjoint to  $i \otimes \mathbf{1}_Z : X \otimes_A Z \rightarrow Y \otimes_A Z$ .

It is obvious that the assertion (ii) is equivalent to the following statement: the operator  $i_*$  maps the closed unit ball in the domain space onto the closed unit ball in the range space. Because of the diagram above, this assertion, in its turn, is equivalent to the statement that  $i^\bullet$  has the same property. But, as an obvious corollary (in fact, an equivalent formulation) of the Hahn-Banach theorem, an adjoint operator has the indicated property if and only if the original operator is isometric. The rest is clear.  $\triangleright$

An immediate corollary is Proposition that was formulated in the beginning of Introduction.

As a byproduct, we have the following observation.

**Proposition 1.2.** *Suppose that  $X, Y, Z$  and  $i$  are as before, and  $Z_0$  is a dense submodule of  $Z$ . Then  $i \otimes_A \mathbf{1}_Z$  is an isometry if and only if the same is true of  $i \otimes_A \mathbf{1}_{Z_0}$ .*

$\triangleleft$  Indeed, the dual modules of  $Z$  and  $Z_0$  coincide, and therefore the assertion (ii) above is valid if and only if it is valid after the replacing of  $Z$  by  $Z_0$ . The rest is clear.  $\triangleright$

Later we shall come across quite a few diagrams like that one above. To write down them all would take too much space. In this connection the following terminology is convenient. We shall say that the morphisms  $\varphi : X_1 \rightarrow X_2$  and  $\psi : Y_1 \rightarrow Y_2$  acting between normed  $A$ -modules, are *isometrically equivalent*, if there exist isometric isomorphisms of  $A$ -modules  $I$  and  $J$  such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ I \downarrow & & \downarrow J \\ Y_1 & \xrightarrow{\psi} & Y_2 \end{array} \quad (1.1)$$

is commutative. In particular, we shall speak about the isometric equivalence of two operators ( $\mathbb{C}$ -modules). As to the isomorphisms  $I$  and  $J$ , we shall say that they *implement* the mentioned kind of the equivalence.

From now on we concentrate on the case  $A := c_0$ . We need some further notation and several elementary facts, concerning  $c_0$ -modules and their tensor products.

Let  $X$  be an arbitrary  $c_0$ -module,  $X_n; n = 1, 2, \dots$  its coordinate submodules (see Introduction). Note that the outer multiplication in  $X_n$  acts as  $\xi \cdot x = \xi_n x$ . We denote by  $\alpha_n^X : X_n \rightarrow X$  the respective natural embeddings, and by  $\beta_n^X : X \rightarrow X_n$  the projections  $x \mapsto x_n$ . Clearly, we have morphisms of  $c_0$ -modules that are isometries and, respectively, coisometries (= quotient maps).

For every  $N = 1, 2, \dots$  we set  $P^N := \sum_{n=1}^N \mathbf{p}^n \in c_0$ . It is easy to see that *for every*  $x \in X_{es}$  (cf. Introduction) *we have*

$$x = \lim_{N \rightarrow \infty} P^N \cdot x. \quad (1.2)$$

Consider the pure algebraic  $c_0$ -module  $\mathbf{X}_{n=1}^\infty X_n$ , consisting of all sequences  $(x_1, \dots, x_n, \dots); x_n \in X_n$  and endowed with the coordinate-wise operations. Introduce the map

$$\sigma^X : X \rightarrow \mathbf{X}_{n=1}^\infty X_n : x \mapsto (x_1, \dots, x_n, \dots);$$

this is, of course, a  $c_0$ -module morphism. Obviously,  $\text{Ker}(\sigma^X)$  coincides with  $\text{Ann}(X)$ , and hence it is closed. Therefore we can (and will) identify the submodule  $\text{Im}(\sigma^X)$  in  $\mathbf{X}_{n=1}^\infty X_n$  with  $X^{\text{red}}$  (cf. Introduction) and endow it with the respective quotient norm.

We see that  $\sigma^X$  is injective if and only if  $X$  is faithful. In particular, if  $X$  is essential and a fortiori faithful (see (1.2)),  $\sigma^X$  is certainly injective.

If  $x \in X_n$ , then the sequence  $\sigma^X(x) = (0, \dots, 0, x, 0, \dots)$  belongs to  $(X^{\text{red}})_n$ . Taking into account that  $\|y\| \geq \|x\|$  for all  $y$  with  $\sigma^X(y) = \sigma^X(x)$ , we immediately obtain

**Proposition 1.3.** *The birestriction  $\sigma_n^X : X_n \rightarrow (X^{\text{red}})_n$  of  $\sigma^X$  is an isometric isomorphism.  $\triangleleft \triangleright$*

**Proposition 1.4.** *For every  $n$ , the  $c_0$ -modules  $(X^*)_n$  and  $(X_n)^*$  are isometrically isomorphic.*

$\triangleleft$  Morphisms  $(\alpha_n^X)^* \alpha_n^{X^*} : (X^*)_n \rightarrow (X_n)^*$  and  $\beta_n^{X^*} (\beta_n^X)^* : (X_n)^* \rightarrow (X^*)_n$  are contractive and inverse to each other.  $\triangleright$

Now let  $Z$  be another  $c_0$ -module. Our object of interest is the  $c_0$ -module  $X \otimes_{c_0} Z$  (cf. Introduction).

Throughout the paper,  $\otimes_p$  will be the symbol of the non-completed projective tensor product of normed spaces ( $= \mathbb{C}$ -modules). The projective tensor norm will be denoted by  $\|\cdot\|_p$ .

**Proposition 1.5.** *There exists an isometric isomorphism  $\rho_n^{X,Z} : X_n \otimes_p Z_n \rightarrow (X \otimes_{c_0} Z)_n$ , well defined by  $x \otimes z \mapsto x \otimes_{c_0} z$ .*

$\triangleleft$  Consider the contractive linear operators  $\rho : X_n \otimes_p Z_n \rightarrow X \otimes_{c_0} Z$  and  $\pi : X \otimes_{c_0} Z \rightarrow X_n \otimes_p Z_n$ , associated with the contractive bilinear operator

$X_n \times Z_n \rightarrow X \otimes_{c_0} Z : (x, z) \mapsto x \otimes_{c_0} z$  and the contractive balanced bilinear operator  $X \times Z \rightarrow X_n \otimes_p Z_n : (x, z) \mapsto \mathbf{p}^n \cdot x \otimes \mathbf{p}^n \cdot z$ , respectively. Since  $\pi \rho = \mathbf{1}$ , we

obtain that  $\rho$  is an isometry (whereas  $\pi$  is a coisometry). Obviously, the image of  $\rho$  is exactly  $(X \otimes_{c_0} Z)_n$ . It remains to denote by  $\rho_n^{X,Z}$  the respective corestriction.  $\triangleright$

Now we turn to the normed module  $(X \otimes_{c_0} Z)^{red}$  and to the coisometric morphism  $\sigma^{X,Z} : X \otimes_{c_0} Z \rightarrow (X \otimes_{c_0} Z)^{red}$ , which is, by definition, the respective corestriction of  $\sigma^{X \otimes Z}_{c_0}$  (cf. above). We want to describe them, up to an isometric isomorphism and, respectively, isometric equivalence, in terms, convenient for their study.

Consider the pure algebraic  $c_0$ -module  $X_{n=1}^\infty(X_n \otimes Z_n)$  with the coordinate-wise operations. For  $x \in X, z \in Z$  we shall denote by  $x \odot z$  the sequence  $(x_1 \otimes z_1, \dots, x_n \otimes z_n, \dots)$ , belonging to this module. Denote by  $X \odot Z$  the submodule of  $X_{n=1}^\infty(X_n \otimes Z_n)$ , defined as the linear span of all such sequences.

Introduce a bilinear operator  $X \times Z \rightarrow X \odot Z : (x, z) \mapsto x \odot z$ ; clearly it is balanced. Therefore it gives rise to the linear operator and, obviously, a surjective  $c_0$ -module morphism  $\odot_{X,Z} : X \otimes_{c_0} Z \rightarrow X \odot Z$ , well defined by  $x \otimes_{c_0} z \mapsto x \odot z$ .

For  $v \in X \odot Z$  we set

$$\|v\|_\odot := \inf \left\{ \sum_{k=1}^m \|x^k\| \|z^k\| \right\}, \quad (1.3)$$

where the infimum is taken over all representations of  $v$  in the form  $\sum_{k=1}^m x^k \odot z^k$ ;  $x^k \in X, z^k \in Z$ .

**Proposition 1.6.** *The function  $v \mapsto \|v\|_\odot$  is a norm on  $X \odot Z$ . Moreover, with respect to this norm  $X \odot Z$  is isometrically isomorphic to  $(X \otimes_{c_0} Z)^{red}$ , and  $\odot_{X,Z}$  is isometrically equivalent to  $\sigma^{X,Z}$ . In more details, there is a commutative diagram*

$$\begin{array}{ccc} X \otimes_{c_0} Z & \xrightarrow{\sigma^{X,Z}} & (X \otimes_{c_0} Z)^{red} \\ \downarrow \mathbf{1} & & \downarrow \iota^{X,Z} \\ X \otimes_{c_0} Z & \xrightarrow{\odot_{X,Z}} & X \odot Z \end{array} \quad (1.4)$$

where  $\iota^{X,Z}$  is an isometric isomorphism of  $c_0$ -modules.

$\triangleleft$  Since  $\odot_{X,Z}$  is surjective,  $X \odot Z$  is a seminormed module with respect to the seminorm  $\|v\|' := \inf \{\|u\|; \odot_{X,Z}(u) = v\}$ .

First, we shall show that  $\|\cdot\|_\odot = \|\cdot\|'$ . Indeed, taking an arbitrary representation  $v = \sum_{k=1}^m x^k \odot z^k$  and looking at  $u := \sum_{k=1}^m x^k \otimes_{c_0} z^k \in X \otimes_{c_0} Z$ , we easily see that  $\|v\|' \leq \|v\|_\odot$ . On the other hand, for every  $\varepsilon > 0$  we can take  $u \in X \otimes_{c_0} Z$  with  $\odot_{X,Z}(u) = v$  and  $\|u\| \leq \|v\|_\odot + \varepsilon$ , and then a representation  $u = \sum_{k=1}^m x^k \otimes_{c_0} z^k$

such that  $\|u\| > \sum_{k=1}^m \|x^k\| \|z^k\| - \varepsilon$ . Since evidently  $v = \sum_{k=1}^m x^k \odot z^k$ , we have  $\|v\|' \geq \|v\|_{\odot} - 2\varepsilon$ , and the reverse inequality follows.

Now take  $u \in X \otimes_{c_0} Z$ . Let  $(\dots, u^n, \dots)$  be the sequence  $\odot_{X,Z}(u)$ . We observe that  $\rho_n^{X,Z}$  takes  $u^n$  to  $u_n := \mathbf{p}^n \cdot u$ ; one can immediately check this on elementary tensors. It easily follows that  $\text{Ker}(\sigma^{X,Z}) = \text{Ker}(\odot_{X,Z})$ . Since both  $\sigma^{X,Z}$  and  $\odot_{X,Z}$  are coisometries, there exists a unique isometric isomorphism  $\iota_{X,Z}$ , making the diagram (1.4) commutative. The rest is clear.  $\triangleright$

Thus, by virtue of Propositions 1.1–1.3, we have, for each  $n$ , a chain of isometric isomorphisms

$$X_n \otimes_p Z_n \xrightarrow{\rho_n^{X,Z}} (X \otimes_{c_0} Z)_n \xrightarrow{\sigma_n^{X,Z}} (X \otimes_{c_0} Z)_n^{\text{red}} \xrightarrow{\iota_n^{X,Z}} (X \odot Z)_n, \quad (1.5)$$

where the last map is the respective birestriction of  $\iota^{X,Z}$ . Denote by  $\varkappa_n^{X,Z} : X_n \otimes_p Z_n \rightarrow (X \odot Z)_n$  their composition. This is, of course, an isometric isomorphism of  $c_0$ -modules, well defined by taking  $x \otimes z$  to  $x \odot z$ .

**Proposition 1.7.** *Suppose that at least one of modules  $X$  and  $Z$  is essential. Then the same is true for  $X \otimes_{c_0} Z$ , and, moreover, for every  $u \in X \otimes_{c_0} Z$ , we have*

$$u = \lim_{N \rightarrow \infty} P^N \cdot u. \quad (1.6)$$

$\triangleleft$  It follows from the equality (1.2), combined with the continuity of the operation  $\otimes_{c_0}$ .  $\triangleright$

From this, taking into account the diagram (1.4), we immediately obtain

**Proposition 1.8.** *If at least one of modules  $X$  and  $Z$  is essential, then  $\odot_{X,Z}$  is an isometric isomorphism of  $c_0$ -modules.*  $\triangleleft \triangleright$

The indicated assumption can not be omitted, even when both of modules are faithful:

**Example 1.9.** Consider  $X := Z := l_\infty$  with the coordinate-wise operations and uniform norm. Take the sequences  $x := (1, 0, 1, 0, 1, 0, \dots) \in X$  and  $z := (0, 1, 0, 1, 0, 1, \dots) \in Z$ . Of course, we have  $\odot_{X,Z}(x \otimes z) = 0$ .

Now take two functionals  $f, g : l_\infty \rightarrow \mathbb{C}$  of norm 1, such that  $f(\xi) = g(\eta) = 0$  for  $\xi, \eta \in c_0$  and  $f(x) = g(z) = 1$ ; these are easily provided by the Hahn-Banach Theorem. Then the bilinear functional  $f \times g : X \times Z \rightarrow \mathbb{C} : (\xi, \eta) \mapsto f(\xi)g(\eta)$  is obviously balanced and contractive. Therefore it gives rise to the contractive functional  $f \otimes g : X \otimes_{c_0} Z \rightarrow \mathbb{C}$ , well defined by  $\xi \otimes \eta \mapsto f(\xi)g(\eta)$ . Since  $(f \otimes g)(x \otimes z) = 1$ , we have  $x \otimes z \neq 0$ . Thus  $\odot_{X,Z}$  is not injective.

Now suppose that we have three  $c_0$ -modules  $X, Y$  and  $Z$ , so far arbitrary, and a bounded  $c_0$ -module morphism  $\varphi : X \rightarrow Y$ . The latter in an obvious way generates the sequence of its birestrictions  $\varphi_n : X_n \rightarrow Y_n$ .

Consider the bounded morphism  $\varphi \otimes \mathbf{1} : X \otimes_{c_0} Z \rightarrow Y \otimes_{c_0} Z$ ; we recall that it is well defined by  $x \otimes_{c_0} z \mapsto \varphi(x) \otimes_{c_0} z$ . Clearly,  $\varphi \otimes \mathbf{1}$  maps  $\text{Ann}(X \otimes_{c_0} Z)$  into  $\text{Ann}(Y \otimes_{c_0} Z)$ . It obviously follows that  $\varphi \otimes \mathbf{1}$  gives rise to the bounded morphism  $(\varphi \otimes \mathbf{1})^{red} : (X \otimes_{c_0} Z)^{red} \rightarrow (Y \otimes_{c_0} Z)^{red}$ , well defined by  $(\varphi \otimes \mathbf{1})^{red}(\sigma^{X,Z}(x \otimes_{c_0} z)) = \sigma^{Y,Z}(\varphi(x) \otimes_{c_0} z); x \in X, z \in Z$ .

Combining this with Proposition 1.6, we obtain the commutative diagram

$$\begin{array}{ccc} X \otimes_{c_0} Z & \xrightarrow{\odot_{X,Z}} & X \odot Z \\ \varphi \otimes \mathbf{1}_{c_0} \downarrow & & \downarrow \varphi \odot \mathbf{1} \\ Y \otimes_{c_0} Z & \xrightarrow{\odot_{Y,Z}} & Y \odot Z \end{array} \quad (1.7)$$

where  $\varphi \odot \mathbf{1}$  is well defined by  $x \odot z \mapsto \varphi(x) \odot z$ . In other words,  $\varphi \odot \mathbf{1}$  takes the sequence  $(\dots, u_n, \dots); u_n \in X_n \otimes_p Z_n$  to the sequence  $(\dots, (\varphi_n \otimes \mathbf{1})u_n, \dots)$ .

Note that we obviously have

$$\|\varphi \odot \mathbf{1}\| \leq \|\varphi \otimes \mathbf{1}_{c_0}\| \leq \|\varphi\| \quad (1.8)$$

Being morphisms of  $c_0$ -modules,  $\varphi \otimes \mathbf{1}$  and  $\varphi \odot \mathbf{1}$  have well defined birestrictions  $(\varphi \otimes \mathbf{1})_n$  and  $(\varphi \odot \mathbf{1})_n$ , respectively, for every  $n$ . Using the identifications, participating in the chain (1.5), for the pairs  $(X, Z)$  and  $(Y, Z)$ , we easily obtain

**Proposition 1.10.** *Both of  $(\varphi \otimes \mathbf{1})_n$  and  $(\varphi \odot \mathbf{1})_n$  are isometrically equivalent to the operator  $\varphi_n \otimes \mathbf{1} : X_n \otimes_p Z_n \rightarrow Y_n \otimes_p Z_n$ .  $\triangleleft \triangleright$*

Finally, Proposition 1.8 immediately implies

**Proposition 1.11.** *Suppose that either both of  $X$  and  $Y$ , or  $Z$  are essential. Then the morphisms  $\varphi \otimes \mathbf{1}_{c_0}$  and  $\varphi \odot \mathbf{1}$  are isometrically equivalent.  $\triangleleft \triangleright$*

## 2. Tensoring injective morphisms

Let  $X, Y, Z$  be normed  $c_0$ -modules,  $\varphi : X \rightarrow Y$  a bounded morphism. Suppose that  $\varphi$  is injective. When we can be sure that  $\varphi \otimes \mathbf{1}_{c_0}$  is also injective? (A kind of a ‘normed’ version of an important typical question in pure algebra).

If we ask the same about  $\varphi \odot \mathbf{1}$ , the situation is clear:

**Proposition 2.1.** *Let  $\varphi$  be injective. Then the same is true with  $\varphi \odot \mathbf{1}$ .*

◁ Together with  $\varphi$ , its birestrictions  $\varphi_n$  are also injective. Then, for pure algebraic reasons, the same is true for operators  $\varphi_n \otimes \mathbf{1} : X_n \otimes_p Z_n \rightarrow Y_n \otimes_p Z_n$ . It remains to recall the way  $\varphi \odot \mathbf{1}$  acts. ▷

From this we obtain

**Proposition 2.2.** *Suppose that  $X$  or  $Z$  is essential. Then, if  $\varphi$  is injective, then the same is true for  $\varphi \otimes_{c_0} \mathbf{1}$ .*

◁ By Propositions 1.8 and 2.1, both  $\sigma^{X,Z}$  and  $\iota^{X,Z}$  in the commutative diagram (1.7) are injective. The rest is clear. ▷

There is another kind of a condition, this time in terms of  $\varphi$  itself, that gives the same result. Suppose that  $\varphi$  is admissible, i.e. it has a left inverse bounded operator (not necessarily morphism of modules). Recall that the Banach algebra  $c_0$  is amenable, and hence every  $c_0$ -module, in particular, our  $Z$ , is flat. This means that for such a  $\varphi$  the morphism  $\varphi \otimes_{c_0} \mathbf{1}$  is not only injective, but topologically injective; see, e.g., [3, Ch.VII]. (Actually, the cited book deals with the “completed” theory, that is with Banach modules and completed module tensor products. But it is easy to observe that the indicated property of  $\varphi \otimes_{c_0} \mathbf{1}$  is valid in the ‘non-completed’ case as well).

However, if we have just an injective morphism between two normed  $c_0$ -modules, let them be even faithful, the situation is different:

**Example 2.3.** Take  $X := Z := l_\infty$  and set  $Y := c_0$ . Consider a sequence  $(\zeta_1, \zeta_2, \dots) \in c_0$  with non-zero terms and introduce  $\varphi : X \rightarrow Y : (\xi_1, \xi_2, \dots) \mapsto (\zeta_1 \xi_1, \zeta_2 \xi_2, \dots)$ . Of course,  $\varphi$  is injective. At the same time, the lower horizontal arrow in (1.7) obviously depicts an injective map whereas the upper arrow, as we know from Example 1.9, does not. Therefore  $\varphi \otimes_{c_0} \mathbf{1}$  can not be injective.

Of course, such a  $\varphi$  is far from to be admissible. But what can happen in the “intermediate” case, when  $\varphi$  is not bound to be admissible, but at least it is topologically injective?

It is easy to show that  $\varphi \otimes_{c_0} \mathbf{1}$  is not bound to be *topologically* injective. Moreover, as the related phenomenon, in the ‘completed’ theory such a morphism is not bound to be even injective (cf. the end of Introduction).

But the present paper deals with the “non-completed” theory, and with a very specific base algebra. It turns out that in such a context we still have a positive result:

**Theorem 2.4.** *Let  $X, Y, Z$  be normed  $c_0$ -modules, and  $\varphi : X \rightarrow Y$  be a topologically injective morphism. Then  $\varphi \otimes \mathbf{1}$  is injective.*

Take  $u \in X \otimes Z; u \neq 0$ ; we want to show that  $(\varphi \otimes \mathbf{1})(u)$  is not 0. If  $\odot_{X,Z}(u) \neq 0$ , that is  $u \notin \text{Ann}(X \otimes Z)$ , then the desired fact follows from Proposition 2.1, combined with the commutative diagram (1.7). Thus we have a right to assume that  $u$  lives in  $\text{Ann}(X \otimes Z)$ .

Consider the quotient maps  $\tau_X : X \rightarrow X_{an}$  and  $\tau_Z : Z \rightarrow Z_{an}$  (cf. Introduction) and set, for brevity,  $\tau := \tau_X \otimes \tau_Z : X \otimes Z \rightarrow X_{an} \otimes Z_{an}$ . Recall that  $X \otimes Z$ , by its definition, is a quotient space of  $X \otimes Z$  (actually, a quotient normed space of  $X \otimes Z$ ) and denote by  $\gamma$  the respective quotient map. It is easy to see that  $\text{Ker}(\tau)$  is the algebraic sum of  $X_{es} \otimes Z$  and  $X \otimes Z_{es}$ . This obviously implies that

$$\gamma(\text{Ker}(\tau)) \subseteq (X \otimes Z)_{es}. \quad (2.1)$$

Fix an arbitrary  $v \in X \otimes Z$  with  $\gamma(v) = u$  and set  $w := \tau(v)$ . We claim that  $w \neq 0$ . Indeed, in the opposite case we have, by (2.1), that  $u \in (X \otimes Z)_{es}$  and hence, by (1.2), that  $u = \lim_{N \rightarrow \infty} P^N \cdot u$ . This, together with  $u \in \text{Ann}(X \otimes Z)$ , gives  $u = 0$ , a contradiction.

Thus  $w$ , being a non-zero vector in  $X_{an} \otimes Z_{an}$ , can be represented as  $w = \sum_{k=1}^n \tilde{x}_k \otimes \tilde{z}_k; \tilde{x}_k \in X_{an}, \tilde{z}_k \in Z_{an}$ , where  $\tilde{x}_1 \neq 0$ , and  $\tilde{z}_k$  are linearly independent.

Take an arbitrary  $x_1 \in X$  such that  $\tau_X(x_1) = \tilde{x}_1$ . Our next claim is that  $\varphi(x_1) \notin Y_{es}$ . Suppose the contrary. Then, by (1.2), we have

$$\varphi(x_1) = \lim_{N \rightarrow \infty} P^N \cdot \varphi(x_1) = \lim_{N \rightarrow \infty} \varphi(P^N \cdot x_1).$$

But this, since  $\varphi$  is topologically injective, implies that  $x_1 = \lim_{N \rightarrow \infty} P^N \cdot x_1$ , that is  $x_1 \in X_{es}$ . Hence we have  $\tilde{x}_1 = 0$ , a contradiction.

This claim implies, by means of a standard corollary of the Hahn-Banach Theorem, that there exists a bounded functional  $f : Y \rightarrow \mathbb{C}$  such that  $f = 0$  on  $Y_{es}$ , and  $f(\varphi(x_1)) = 1$ . The same corollary provides a bounded functional  $\tilde{g} : Z_{an} \rightarrow \mathbb{C}$  such that  $\tilde{g}(\tilde{z}_1) = 1$  and  $\tilde{g}(\tilde{z}_k) = 0$  for  $k = 2, \dots, n$ . Take an arbitrary  $z_k \in Z$  with  $\tau_Z(z_k) = \tilde{z}_k; k = 1, \dots, n$  and consider the bounded functional  $g := \tilde{g}\tau_Z : Z \rightarrow \mathbb{C}$ . Then we have, of course, that  $g(z_1) = 1$  and  $g(z_k) = 0$  for  $k = 2, \dots, n$ .

Now introduce the bounded bilinear functional  $f \times g : Y \times Z \rightarrow \mathbb{C} : (y, z) \mapsto f(y)g(z)$ . Since  $f = 0$  on  $Y_{es}$  and  $g = 0$  on  $Z_{es}$ , it is evidently balanced. Therefore it gives rise to the bounded linear functional, say  $h : Y \otimes Z \rightarrow \mathbb{C}$ , well defined by  $h(y \otimes z) = f(y)g(z)$ .

We easily see that  $h = 0$  on  $(Y \otimes_{c_0} Z)_{es}$ . At the same time the element  $v - \sum_{k=1}^n x_k \otimes z_k$  belongs to  $\text{Ker}(\tau)$ . Therefore we have, by (2.1), that  $u - \sum_{k=1}^n x_k \otimes z_k \in (X \otimes_{c_0} Z)_{es}$ , and consequently  $(\varphi \otimes \mathbf{1})(u) - \sum_{k=1}^n \varphi(x_k) \otimes z_k$  lies in  $(Y \otimes_{c_0} Z)_{es}$ . Therefore  $h(\varphi \otimes \mathbf{1}(u)) = h(\sum_{k=1}^n \varphi(x_k) \otimes z_k)$ , and the latter number is, of course, 1. It follows that  $(\varphi \otimes \mathbf{1})(u) \neq 0$ .  $\triangleright$

### 3. Tensoring isometric morphisms

In this section we shall deal with homogeneous  $c_0$ -modules, defined in Introduction. It is a rather large class of normed  $c_0$ -modules. In particular, we have

**Proposition 3.1.** *Suppose that  $X$  is an essential normed  $c_0$ -module, consisting of some complex-valued sequences and endowed with the coordinate-wise outer multiplication. Then  $X$  is homogeneous.*

$\triangleleft$  If  $x, y \in X, x = (\dots, \lambda_n, \dots), y = (\dots, \mu_n, \dots); \lambda_n, \mu_n \in \mathbb{C}$ , then the equalities  $\|x_n\| = \|y_n\|; n = 1, 2, \dots$  mean, of course, just that  $|\lambda_n| = |\mu_n|$ . Therefore, for every  $N \in \mathbb{N}$  we have  $P^N \cdot x = \xi \cdot P^N \cdot y$  for some  $\xi = (\dots, \xi_n, \dots) \in c_0$  such that  $|\xi_n| = 1$  provided  $n \leq N$  and  $\xi_n = 0$  otherwise. It follows that  $\|P^N \cdot x\| \leq \|P^N \cdot y\|$ , and similarly the reverse inequality is valid. But, since  $X$  is essential, we can use (2). The rest is clear.  $\triangleright$

Note a useful

**Proposition 3.2.** *Let  $X$  be a homogeneous  $c_0$ -module,  $x \in X_{es}$  and  $y \in X$ . Suppose that  $\|x_n\| \leq \|y_n\|$  for all  $n$ . Then  $\|x\| \leq \|y\|$ .*

$\triangleleft$  We have  $\|x_n\| = \xi_n \|y_n\|$  for some  $0 \leq \xi_n \leq 1; n = 1, 2, \dots$ . Fix, for a moment,  $N$ , and consider  $\xi := (\xi_1, \dots, \xi_N, 0, 0, \dots) \in c_0$ . Then, by homogeneity, we have  $\|P^N \cdot x\| = \|\xi P^N \cdot y\| \leq \|y\|$ . It remains to use (1.2)  $\triangleright$

**Proposition 3.3.** *Let  $Z$  be a  $c_0$ -module. Assume that, for every essential homogeneous  $c_0$ -modules  $X$  and  $Y$  and an isometric morphism  $i : X \rightarrow Y$  the morphism  $i \otimes \mathbf{1} : X \otimes_{c_0} Z \rightarrow Y \otimes_{c_0} Z$  is also isometric. Then, for every  $n = 1, 2, \dots$ , the coordinate submodule  $Z_n$  is, up to an isometric isomorphism of normed spaces, a dense subspace of  $L_1(\Omega_n, \mu_n)$  for some measure space  $(\Omega_n, \mu_n)$ .*

$\triangleleft$  Suppose that, for a certain  $n$ ,  $Z_n$  does not satisfy the indicated condition. Then it easily follows from the criterion of Grothendieck [5, Thm. 1] that there are normed spaces  $X, Y$  and an isometric operator  $i : X \rightarrow Y$  such that the operator  $i \otimes \mathbf{1} : X \otimes_p Z_n \rightarrow Y \otimes_p Z_n$  fails to be an isometry.

Set, for every  $\xi = (\xi_1, \dots, \xi_n, \dots) \in c_0, x \in X, y \in Y, \xi \cdot x := \xi_n x$  and  $\xi \cdot y := \xi_n y$ . In this way we obviously make  $X$  and  $Y$   $c_0$ -modules that are essential and

homogeneous. Moreover,  $i$  becomes a  $c_0$ -module morphism. Since  $X$  and  $Y$  are essential, it is sufficient, by virtue of Proposition 1.11, to show that the operator  $i \odot \mathbf{1} : X \odot Z \rightarrow Y \odot Z$  is not an isometry.

We see that, for  $m \neq n$ , we have  $X_m = Y_m = 0$ . It easily follows that  $X \odot Z = (X \odot Z)_n$  and  $Y \odot Z = (Y \odot Z)_n$ . Therefore the isometric isomorphisms  $\mathcal{K}_n^{X,Z}$  and  $\mathcal{K}_n^{Y,Z}$  (see Section 1) act between  $X_n \otimes_p Z_n$  and  $X \odot Z$ , and, respectively, between  $Y_n \otimes_p Z_n$  and  $Y \odot Z$ . Moreover, these isometric isomorphisms obviously implement an isometric equivalence of the operators  $i \otimes \mathbf{1}$  and  $i \odot \mathbf{1}$  (cf. (1.1)). Consequently, since the former of these two is not an isometry, the same is true for the latter.  $\triangleright$

Our principal aim is to show that the converse statement is valid. Actually, we shall prove a slightly stronger assertion.

The main step in our proof is the following technical lemma. In what follows  $S$  is an arbitrary homogeneous normed  $c_0$ -module with the following properties:

(i) there exists a natural  $N$  such that  $S$ , up to a linear isomorphism, is  $\bigoplus_{n=1}^N S_n$ . (In other words, for every  $x \in S$  we have  $P^N \cdot x = x$ ).

(ii) for every  $n = 1, \dots, N$ ,  $S_n$  is a normed subspace of  $L_1(\Omega_n, \mu_n)$  for some measure space  $(\Omega_n, \mu_n)$ , consisting of all step functions (= linear combinations of characteristic functions of  $\mu_n$ -measurable subsets in  $\Omega_n$ ).

**Lemma 3.4.** *Let  $X, Y$  be normed homogeneous  $c_0$ -modules and  $i : X \rightarrow Y$  a morphism. Suppose we are given  $u \in X \odot S$ . Let  $v := (i \odot \mathbf{1}_S)(u) \in Y \odot S$  be represented as  $v = \sum_{k=1}^m y^k \odot g^k$ ;  $y^k \in Y, g^k \in S$ . Then for every  $n = 1, \dots, N$  there exist natural number  $M$ ,  $x^{kl} \in X_n$  and  $g^{kl} \in S$ ;  $k = 1, \dots, m, l = 1, \dots, M$  such that for*

$$y^{kl} := y_1^k + y_2^k + \dots + y_{n-1}^k + i_n(x_n^{kl}) + y_{n+1}^k + \dots + y_N^k \quad (3.1)$$

we have

$$v = \sum_{k=1}^m \sum_{l=1}^M y^{kl} \odot g^{kl} \quad (3.2)$$

and

$$\sum_{k=1}^m \sum_{l=1}^M \|y^{kl}\| \|g^{kl}\| \leq \sum_{k=1}^m \|y^k\| \|g^k\|. \quad (3.3)$$

$\triangleleft$  Let  $\sum_{s=1}^{m'} x^s \odot f^s$  be an arbitrary representation of  $u$ . Remembering, what  $S_n$  is, we can find  $M \in \mathbb{N}$  and a partition  $\Omega_n = \sqcup_{l=1}^M \Delta_l$ , where  $\Delta_l$ ;  $l = 1, \dots, M$  are  $\mu_n$ -measurable subsets of  $\Omega_n$  such that all  $g_n^k, f_n^s$  are constant functions on each  $\Delta_l$ . In particular, for every  $k = 1, \dots, m$ ,  $g_n^k$  has the form  $\sum_{l=1}^M \lambda^{kl} \chi_l$ , where  $\lambda^{kl} \in \mathbb{C}$  and  $\chi_l$  is the characteristic function of  $\Delta_l$ .

Now for every  $k = 1, \dots, m, l = 1, \dots, M$  we set

$$g^{kl} := \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g_1^k + \dots + \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g_{n-1}^k + \lambda^{kl}\chi_l + \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g_{n+1}^k + \dots + \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g_N^k \quad (3.4)$$

Since  $\|\lambda^{kl}\chi_l\| = \|\frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g_n^k\|$  and  $S$  is homogeneous, we see that

$$\|g^{kl}\| = \|\frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g^k\|$$

for all  $k, l$ . But, living in  $L_1(\cdot)$ , we have  $\sum_{l=1}^M \|\lambda^{kl}\chi_l\| = \|g_n^k\|$ . Therefore for all  $k$  we have  $\sum_{l=1}^M \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} = 1$ . Hence  $g^k = \sum_{l=1}^M g^{kl}$  and

$$\|g^k\| = \sum_{l=1}^M \|\frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|}g^k\| = \sum_{l=1}^M \|g^{kl}\|.$$

From this we have

$$v = \sum_{k=1}^m \sum_{l=1}^M y^k \odot g^{kl} \quad \text{and} \quad \sum_{k=1}^m \sum_{l=1}^M \|y^k\| \|g^{kl}\| = \sum_{k=1}^m \|y^k\| \|g^k\|. \quad (3.5)$$

Let us concentrate on  $v_n$ . It follows from (3.5) and (3.4) that

$$v_n = \sum_{k=1}^m \sum_{l=1}^M y_n^k \otimes \lambda^{kl}\chi_l = \sum_{l=1}^M (\sum_{k=1}^m \lambda^{kl}y_n^k) \otimes \chi_l \quad (3.6)$$

But, as we remember,  $v = (i \odot \mathbf{1}_S)(u)$ , and  $u$  has the representation, indicated above. Therefore we have  $v = \sum_{s=1}^{m'} i('x^s) \odot f^s$ . Besides, by the choice of  $\Delta_l$ , we have, for all  $s$ , that  $f_n^s = \sum_{l=1}^M \nu^{sl}\chi_l$  for some  $\nu^{sl} \in \mathbb{C}$ . Thus

$$v_n = \sum_{l=1}^M (\sum_{s=1}^m \nu^{sl} i_n('x^s)) \otimes \chi_l = \sum_{l=1}^M i_n(x^l) \otimes \chi_l, \quad (3.7)$$

where we set  $x^l := \sum_{s=1}^m \nu^{sl}('x^s)$ .

But  $\chi_l; l = 1, \dots$ , are linearly independent in  $S_n$ . Thus, comparing (3.7) and (3.6), we see that

$$\sum_{k=1}^m \lambda^{kl}y_n^k = i_n(x^l) \quad \text{for all } l. \quad (3.8)$$

Now introduce numbers

$$\alpha^{kl} := (\lambda^{kl})^{-1} \frac{\|\lambda^{kl}y_n^k\|}{\sum_{t=1}^m \|\lambda^{tl}y_n^t\|} \quad \text{provided } \lambda^{kl} \neq 0 \quad \text{and} \quad \alpha^{kl} := 0 \quad \text{otherwise.}$$

Finally, set  $x_n^{kl} := \alpha_n^{kl} x^l$ .

Take  $y^{kl}$  as in (3.1). Look at  $v' := \sum_{k=1}^m \sum_{l=1}^M y^{kl} \odot g^{kl}$ . By (3.1) and (3.5),  $v'_{n'} = v_{n'}$  for all  $n' \neq n$ . As to  $v'_n$ , it is equal to

$$\begin{aligned} \sum_{k=1}^m \sum_{l=1}^M y_n^{kl} \otimes g_n^{kl} &= \sum_{l=1}^M \sum_{k=1}^m i_n(x^{kl}) \otimes \lambda^{kl} \chi_l = \sum_{l=1}^M \sum_{k=1}^m i_n(\alpha^{kl} \lambda^{kl} x_n^l) \otimes \chi_l = \\ &= \sum_{l=1}^M \sum_{k=1}^m i_n \left( \frac{\|\lambda^{kl} y_n^k\|}{\sum_{t=1}^m \|\lambda^{tl} y_n^t\|} x^l \right) \otimes \chi_l = \sum_{l=1}^M i_n(x^l) \otimes \chi_l = \sum_{l=1}^M \lambda^{kl} y_n^l \otimes \chi_l, \end{aligned}$$

that is, by (3.8), to  $v_n$ . Thus  $v$  and  $v'$  have the same coordinates and hence, since  $Y \odot S$  is essential, they coincide. The equality (3.2) follows.

It remains to obtain (3.3). For this, we want to show that for all  $l$  we have

$$\|i_n(x_n^{kl})\| \leq \|y_n^k\|. \quad (3.9)$$

If  $\alpha^{kl} = 0$ , this is immediate. Otherwise we have

$$\|i_n(x_n^{kl})\| = \|\alpha^{kl} i_n(x^l)\| = \left\| \frac{\|y_n^k\|}{\sum_{t=1}^m \|\lambda^{tl} y_n^t\|} \left( \sum_{t=1}^m \lambda^{tl} y_n^t \right) \right\|.$$

and (3.9) follows from the triangle inequality for norms.

Now it is time to use that  $Y$  (not only  $S$ ) is homogeneous. We have just shown that  $\|y_n^{kl}\| \leq \|y_n^k\|$ , and, of course, we have  $\|y_n^{kl}\| = \|y_{n'}^k\|$  for all  $n' \neq n$ . Therefore, since all  $y^{kl}$  belong to  $Y_{es}$ , Proposition 3.2 implies that

$$\|y^{kl}\| \leq \|y^k\|.$$

Consequently, we have

$$\sum_{k=1}^m \sum_{l=1}^M \|y^{kl}\| \|g^{kl}\| \leq \sum_{k=1}^m \sum_{l=1}^M \|y^k\| \|g^{kl}\|,$$

and, because of (3.5), we are done.  $\triangleright$

**Lemma 3.5.** *Let  $X, Y, S$  be as in the previous lemma, and  $i : X \rightarrow Y$  an isometric morphism. Then the morphism  $i \otimes \mathbf{1}_S : X \otimes_{c_0} S \rightarrow Y \otimes_{c_0} S$  is also isometric.*

$\triangleleft$  Of course,  $S$  is essential. Therefore, by virtue of Proposition 1.11, it is sufficient to prove that the morphism  $i \odot \mathbf{1}_S : X \odot S \rightarrow Y \odot S$  is isometric.

Fix an arbitrary  $u \in X \odot S$  and set  $v := (i \odot \mathbf{1})(u) \in Y \odot S$ . Our task is to show that  $\|u\| = \|v\|$ .

Take an arbitrary representation  $v = \sum_{k=1}^m y^k \odot g^k; g^k \in S$ . Set in the previous lemma  $n := 1$ . Getting rid of double sums, we can say that this lemma gives us a representation

$$v = \sum_{k=1}^{m_1} y^{1k} \odot g^{1k},$$

where, for some  $x_1^{1k} \in X_1, k = 1, \dots, m_1$  and  $y_s^{1k}, s = 2, \dots, N$  we have

$$y^{1k} = i_1(x_1^{1k}) + y_2^{1k} + y_3^{1k} + \dots + y_N^{1k}$$

and

$$\sum_{k=1}^{m_1} \|y^{1k}\| \|g^{1k}\| \leq \sum_{k=1}^m \|y^k\| \|g^k\|.$$

Now apply Lemma 3.4 to the just obtained representation of  $v$  and  $n := 2$ . Looking at the form of the relevant  $y^{kl}$  in the situation when the role of  $y^k$  is played by  $y^{1k}$  and again getting rid of double sums, we obtain a representation

$$v = \sum_{k=1}^{m_2} y^{2k} \odot g^{2k},$$

where, for some  $x_1^{2k} \in X_1, x_2^{2k} \in X_2, k = 1, \dots, m_2$  and  $y_s^{2k}, s = 3, \dots, N$ , we have

$$y^{2k} = i_1(x_1^{2k}) + i_2(x_2^{2k}) + y_3^{2k} + \dots + y_N^{2k},$$

and

$$\sum_{k=1}^{m_1} \|y^{2k}\| \|g^{2k}\| \leq \sum_{k=1}^{m_1} \|y^{1k}\| \|g^{1k}\| \quad (\text{and hence} \leq \sum_{k=1}^m \|y^k\| \|g^k\|).$$

After this we apply Lemma 3.4 to this latter representation of  $v$  and  $n := 3$ , and so on. On the  $N$ th step, again (the last time) getting rid of double sums, we come to a representation of  $v$  as

$$v = \sum_{k=1}^{m_N} y^{Nk} \odot g^{Nk},$$

where, for some  $x_1^{Nk} \in X_1, x_2^{Nk} \in X_2, \dots, x_N^{Nk} \in X_N; k = 1, \dots, m_N$  we have

$$y^{Nk} = i_1(x_1^{Nk}) + i_2(x_2^{Nk}) + \dots + i_N(x_N^{Nk})$$

and

$$\sum_{k=1}^{m_N} \|y^{Nk}\| \|g^{Nk}\| \leq \sum_{k=1}^m \|y^k\| \|g^k\|.$$

Now introduce  $x^k := x_1^{Nk} + \dots + x_N^{Nk} \in X; k = 1, \dots, m_N$ . Obviously,  $y^{Nk} = i(x^k)$  and hence  $i \odot \mathbf{1}_S(\sum_{k=1}^{m_N} x^k \odot g^{Nk}) = v$ . But  $i \odot \mathbf{1}_S$  is injective (see Proposition 2.1). Therefore  $\sum_{k=1}^{m_N} x^k \odot g^{Nk}$  is exactly  $u$ . Recalling that  $i$  is isometric, we have

$$\|u\| \leq \sum \|x^k\| \|g^{Nk}\| = \sum \|y^{Nk}\| \|g^{Nk}\|,$$

and hence

$$\|u\| \leq \sum_{k=1}^m \|y^k\| \|g^k\|.$$

Taking the respective infimum in the expression (1.3) for the norm  $\|\cdot\|_\odot$ , we have the estimate  $\|u\| \leq \|v\|$ . Since, by (1.8),  $i \odot \mathbf{1}$  is contractive, the desired equality follows.  $\triangleright$

**Lemma 3.6.** *The assertion of the previous lemma remains true, if we replace the module  $S$  by an arbitrary module  $Z$  such that*

- (i) *there exists a natural  $N$  such that  $Z$  is linearly isomorphic to  $\bigoplus_{n=1}^N Z_n$ .*
- (ii) *for every  $n = 1, \dots, N$ ,  $Z_n$  is, up to an isometric isomorphism, a dense normed subspace of  $L_1(\Omega_n, \mu_n)$  for some measure space  $(\Omega_n, \mu_n)$ .*

$\triangleleft$  Denote by  $\bar{Z}$  and  $\bar{Z}_n; n = 1, \dots, N$  the completions of the  $c_0$ -modules  $Z$  and  $Z_n$ , respectively.

Take  $z \in Z$ . Obviously, we have

$$\max\{\|z_n\|; n = 1, \dots, N\} \leq \|z\| \leq \sum_{n=1}^N \|z_n\|.$$

Therefore a sequence  $z^m$  is a Cauchy sequence in  $Z$  if and only if for every  $n = 1, \dots, N$  the sequence  $z_n^m$  is a Cauchy sequence in  $Z_n$ . It easily follows that  $\bar{Z}$  is isometrically isomorphic to the algebraic direct sum  $\bigoplus_{n=1}^N \bar{Z}_n$ , endowed with the norm, well defined by  $\|z\| = \lim_{m \rightarrow \infty} \|z^m\|$ , where  $z^m$  is an arbitrary sequence in  $Z$  such that  $\lim_{m \rightarrow \infty} z_n^m = z_n$  for every  $n$ . Obviously,  $\bar{Z}_n$  is isometrically isomorphic to the space  $L_1(\Omega_n, \mu_n)$ , mentioned in the formulation. It easily follows that  $\bar{Z}$  contains the dense submodule  $S$ , satisfying the condition of Lemma 3.5. By virtue of that lemma,  $i \otimes_{c_0} \mathbf{1}_S$  is an isometry.

Therefore, by Proposition 1.2, the same is true for  $i \otimes_{c_0} \mathbf{1}_{\bar{Z}}$ , and this, in its turn, gives the desired property of  $i \otimes_{c_0} \mathbf{1}_Z$ .  $\triangleright$

**Theorem 3.7.** *Let  $Z$  be a homogeneous  $c_0$ -module, satisfying the condition (ii) of the previous lemma. Further, let  $X$  and  $Y$  be two other homogeneous  $c_0$ -modules,  $i: X \rightarrow Y$  an isometric morphism. Suppose that at least one of modules  $X$  and  $Z$  is essential. Then the morphism  $i \otimes_{c_0} \mathbf{1}: X \otimes_{c_0} Z \rightarrow Y \otimes_{c_0} Z$  is also isometric.*

◁ Take  $u \in X \otimes_{c_0} Z$ . Our task is to show that  $\|i \odot \mathbf{1}_Z(u)\| = \|u\|$ .

Fix, for a time,  $N \in \mathbb{N}$  and denote by  $Z^N$  the submodule  $\{P^N \cdot u; u \in Z\}$  of  $Z$ . Consider the diagram

$$\begin{array}{ccc} X \otimes_{c_0} Z^N & \xrightarrow{\mathbf{1}_X \otimes i} & X \otimes_{c_0} Z \\ \downarrow i' & & \downarrow i' \\ Y \otimes_{c_0} Z^N & \xrightarrow{\mathbf{1}_Y \otimes i} & Y \otimes_{c_0} Z \end{array}$$

where  $i' := i \otimes \mathbf{1}_Z$ , and  $i : Z^N \rightarrow Z$  is the natural embedding. By Lemma 3.6, the left vertical arrow depicts an isometric morphism. Further,  $\mathbf{1}_X \otimes i$  is contractive and has a contractive right inverse, namely  $\mathbf{1}_X \otimes j$ , where  $j : Z \rightarrow Z^N$  acts as  $z \mapsto P^N \cdot z$ . Therefore  $\mathbf{1}_X \otimes i$  is an isometry, and the same is true with  $\mathbf{1}_Y \otimes i$ .

For every  $x \in X$  and  $z \in Z$  we have  $P^N \cdot (x \otimes_{c_0} z) = x \otimes_{c_0} P^N \cdot z$ . From this, representing  $u$  as a sum of elementary tensors, we see that  $P^N \cdot u = (\mathbf{1}_X \otimes i)(v)$  for some  $v \in X \otimes_{c_0} Z^{(N)}$ . Therefore, since our diagram is obviously commutative and its three morphisms, mentioned above, are isometries, we have

$$\|(i \otimes \mathbf{1}_Z)(P^N \cdot u)\| = \|P^N \cdot u\|.$$

Now observe that, by Proposition 1.7, we have  $u = \lim_{N \rightarrow \infty} P^N \cdot u$ , and hence  $\|(i \odot \mathbf{1}_Z)(u)\| = \lim_{N \rightarrow \infty} \|(i \odot \mathbf{1}_Z)(P^N \cdot u)\|$ . The rest is clear. ▷

Combining this theorem with Proposition 3.3, we immediately obtain Theorem I, formulated in Introduction, with its mentioned corollaries for sequence modules and some other modules.

From this theorem, in its turn, a Hahn-Banach type theorem, formulated in Introduction as Theorem II, easily follows. Indeed, it is a well known fact that, for a normed space  $E$ , its dual space is isometrically isomorphic to  $L_\infty(\Omega, \mu)$  for some measure space  $(\Omega, \mu)$  if and only if  $E$  is isometrically isomorphic to a dense subspace of  $L_1(\Omega, \mu)$ . (‘If’ part is the classics. To obtain the ‘only if’ part we can recall, for example, that  $L_\infty(\Omega, \mu)$ , being a von Neumann algebra, has only one, up to an isometric isomorphism, Banach predual space; cf., e.g., [14, Cor.III.3.9]). Therefore, if we take this fact into account, Theorem II immediately follows from Theorem I, combined with Propositions 1.1 and 1.4.

#### 4. A counter-example

Here we want to show that the conditions in our main theorem, concerning the property of modules to be essential, can not be omitted, even within the class of faithful homogeneous modules. Namely, we shall show that the module  $l_\infty$  (apparently the first faithful non-essential module that comes in mind), is not extremely flat with respect to the mentioned class.

At first let us make some observations of general character.

Let  $X$  be a  $c_0$ -module. A subset  $M$  of  $\mathbb{N}$  is called a *support* of  $X$ , if we have  $X_n = 0$  for all  $n \notin M$ .

**Lemma 4.1.** *Let  $X$  and  $Z$  be two modules that have non-intersecting supports. Then for every  $x \in X, x' \in X_{es}, z \in Z, z' \in Z_{es}$  we have  $x' \underset{c_0}{\otimes} z = x \underset{c_0}{\otimes} z' = 0$  in  $X \underset{c_0}{\otimes} Z$ .*

◁ By (1.2), we have

$$x' \underset{c_0}{\otimes} z = \lim_{N \rightarrow \infty} P^N \cdot x' \underset{c_0}{\otimes} z = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mathbf{p}^n \cdot x' \underset{c_0}{\otimes} \mathbf{p}^n \cdot z.$$

But the condition on supports implies that, for every  $n$ , either  $\mathbf{p}^n \cdot x'$  or  $\mathbf{p}^n \cdot z$  is 0. The rest is clear. ▷

For  $x \in X$ , we shall denote by  $\tilde{x}$  the coset  $x + X_{es} \in X_{an}$ .

**Proposition 4.2.** *Let  $X$  and  $Z$  be as before. Then there exists the isometric isomorphism of normed spaces  $I_{X,Z} : X \underset{c_0}{\otimes} Z \rightarrow X_{an} \underset{p}{\otimes} Z_{an}$ , well defined by*

$$x \underset{c_0}{\otimes} z \mapsto \tilde{x} \underset{p}{\otimes} \tilde{z}.$$

◁ Consider the bilinear operator  $X \times Z \rightarrow X_{an} \underset{p}{\otimes} Z_{an} : (x, z) \mapsto \tilde{x} \underset{p}{\otimes} \tilde{z}$ ; it is obviously contractive and balanced. Therefore it gives rise to a contractive operator  $I_{X,Z}$ , well defined as it was indicated.

Take  $v \in X_{an} \underset{p}{\otimes} Z_{an}$ , represented, say, as  $\sum_{k=1}^n \tilde{x}_k \underset{p}{\otimes} \tilde{z}_k; x_k \in X, z_k \in Z$ . Then we have  $v = I_{X,Z}(u)$ , where  $u = \sum_{k=1}^n x_k \underset{c_0}{\otimes} z_k$  with arbitrary  $x_k, z_k$ , taken in the respective cosets. Obviously, what we have to do is to show that  $\|u\| \leq \|v\|$ .

Take some  $x'_k \in X_{es}, z'_k \in Z_{es}$ . Lemma 4.1 implies that

$$u = \sum_{c_0} (x_k + x'_k) \underset{c_0}{\otimes} (z_k + z'_k). \text{ Therefore } \|u\| \leq \sum_{k=1}^n \|x_k + x'_k\| \|z_k + z'_k\|.$$

Since  $x'_k, z'_k$  can be chosen in an arbitrary way, we have  $\|u\| \leq \sum_{k=1}^n \|\tilde{x}_k\| \|\tilde{z}_k\|$ . Finally, since the taken representation of  $v$  is also arbitrary, the very definition of the projective tensor norm gives the desired inequality. ▷

Now consider the normed quotient space ('ultraproduct')  $l_\infty/c_0$ . Since it is not isometrically isomorphic to any space of the class  $L_1(\Omega, \mu)$ , the theorem of Grothendieck, cited in Introduction, implies that there exist normed spaces  $E, F$  and an isometric operator  $\tilde{i} : E \rightarrow F$  such that the operator

$$\tilde{i} \otimes_p \mathbf{1} : E \otimes_p (l_\infty/c_0) \rightarrow F \otimes_p (l_\infty/c_0)$$

is not an isometry. Let us choose and fix these  $E, F$  and  $\tilde{i}$ .

In what follows, we shall need, apart from the already used tensor product ' $\otimes_p$ ', the non-completed *injective* tensor product of normed spaces and bounded operators, denoted by ' $\otimes_i$ ' (see, e.g., [15, Ch.I.4] or [16, Ch.3]). The injective tensor norm will be denoted by  $\|\cdot\|_i$ .

Consider the normed space  $l_\infty \otimes_i E$ . Evidently, it is a  $c_0$ -module with the outer multiplication well defined by  $\xi \cdot (\eta \otimes x) := \xi\eta \otimes x; \xi \in c_0, \eta \in l_\infty, x \in E$ .

This module is *contractive*: if  $m_\xi : l_\infty \rightarrow l_\infty$  acts as  $\eta \mapsto \xi\eta$ , then, for every  $u \in l_\infty \otimes_i E$ , we have  $\xi \cdot u = (m_\xi \otimes_i \mathbf{1}_E)(u)$ , and hence

$$\|\xi \cdot u\|_i \leq \|m_\xi \otimes_i \mathbf{1}_E\| \|u\| \leq \|m_\xi\| \|\mathbf{1}_E\| \|u\| \leq \|\xi\| \|u\|.$$

Besides, the introduced module is also *faithful*. Indeed, if  $u \in l_\infty \otimes_i E$  is not 0, then it has a representation  $u = \sum_{k=1}^n \eta^k \otimes x^k$ , where  $x^k$  are linearly independent and  $\eta^1 \neq 0$ . Take  $\xi \in c_0$  with  $\xi \cdot \eta^1 \neq 0$  and  $g \in E^*$  with  $g(x^1) \neq 0, g(x^2) = \dots = g(x^n) = 0$ . Then  $m_\xi \otimes_i g : l_\infty \otimes_i E \rightarrow l_\infty \otimes_i \mathbb{C} = l_\infty$  takes  $\xi \cdot u$  to  $\xi\eta^1 \otimes 1 = \xi\eta^1$ . Therefore  $\xi \cdot u \neq 0$ .

Finally, the module  $l_\infty \otimes_i E$  is *homogeneous*. This fact can be deduced from the known properties of the operation  $\otimes_i C(\Omega)$  (see, e.g., *idem*) and the identification of  $l_\infty$  with  $C(\beta\mathbb{N})$ . But we prefer to give a simpler proof.

Obviously, it suffices to show that for  $u \in l_\infty \otimes_i E; u = \sum_{k=1}^n \xi^k \otimes x^k$  we have

$$\|u\|_i = \sup\{\|\mathbf{p}^n \cdot u\|_i; n = 1, 2, \dots\}.$$

Take  $f \in (l_\infty)^*$  and  $g \in E^*$  with  $\|f\| = \|g\| = 1$ . Then we have  $(f \otimes g)(u) = f(\eta^g)$ , where  $\eta^g := \sum_{k=1}^n g(x^k) \xi^k$ . Hence  $|(f \otimes g)(u)| \leq \|\eta^g\| = \sup\{|(\eta^g)_n|; n = 1, 2, \dots\}$ . But for every  $n$  we have

$$|(\eta^g)_n| = \left\| \sum_{k=1}^n \mathbf{p}^n \xi^k g(x^k) \right\| = \|(\mathbf{1} \otimes g)(\sum_{k=1}^n \mathbf{p}^n \xi^k \otimes x^k)\| = \|(\mathbf{1} \otimes g)(\mathbf{p}^n \cdot u)\| \leq \|\mathbf{p}^n \cdot u\|.$$

Therefore the number  $\|u\|_i$ , which is, by definition,  $\sup\{|(f \otimes g)(u)|; f \in (l_\infty)^*, g \in E^*; \|f\| = \|g\| = 1\}$ , does not exceed  $\sup\{\|\mathbf{p}^n \cdot u\|_i; n = 1, 2, \dots\}$ . Since the reverse inequality is obvious, we are done.

In the same way we define the contractive faithful homogeneous  $c_0$ -module  $l_\infty \otimes_i F$ . Finally, consider the operator  $\mathbf{i} := \mathbf{1} \otimes_i \tilde{i} : l_\infty \otimes_i E \rightarrow l_\infty \otimes_i F$ , which is evidently a morphism of  $c_0$ -modules. Because of the injective property of the operation ‘ $\otimes_i$ ’ (see, e.g., [15, Ch.I.4.3] or [16, p. 47]),  $\mathbf{i}$  is an isometry.

From now on, it is convenient for us to use the notation  $X$  for  $l_\infty \otimes_i E$  and  $Y$  for  $l_\infty \otimes_i F$ .

**Theorem 4.3.** *The morphism  $\mathbf{i} \otimes \mathbf{1} : X \otimes_{c_0} l_\infty \rightarrow Y \otimes_{c_0} l_\infty$  is not an isometry. As a corollary, the module  $l_\infty$  is not extremely flat with respect to the class of all homogeneous normed  $c_0$ -modules.*

◁ We shall write  $Z$  instead of  $l_\infty$ , and just  $\mathbf{1}$  instead of  $\mathbf{1}_Z$ . Note that we have  $Z_{an} = l_\infty/c_0$ .

Denote by  $Z^{od}$  and  $Z^{ev}$  the submodules of  $Z$ , consisting of sequences with the zero even terms and, respectively, zero odd terms. Besides, denote by  $\mathbf{1}_{an}$  and  $\mathbf{1}_\bullet$  the identity operators on  $Z_{an}$  and, respectively, on  $(Z^{ev})_{an}$ . Our first claim is

1<sup>0</sup>. The operator  $\tilde{i} \otimes_{c_0} \mathbf{1}_\bullet : E \otimes_p (Z^{ev})_{an} \rightarrow F \otimes_p (Z^{ev})_{an}$  is not an isometry.

Indeed, taking the sequence  $(0, \xi_2, 0, \xi_4, 0, \dots)$  to  $(\xi_2, \xi_4, \dots)$ , we obtain isometric isomorphisms of normed spaces (by no means of modules)  $j : Z^{ev} \rightarrow Z$ ,  $j_{es} : (Z^{ev})_{es} \rightarrow Z_{es} = c_0$  and, passing to respective cosets,  $j_{an} : (Z^{ev})_{an} \rightarrow Z_{an}$ . Then we easily see that the operators  $\tilde{i} \otimes_p \mathbf{1}_\bullet$  and  $\tilde{i} \otimes_p \mathbf{1}_{an}$  are isometrically equivalent. The rest is clear.

From now on we shall use the brief notation  $X^{od}$  for  $Z^{od} \otimes_i E$ ,  $Y^{od}$  for  $Z^{od} \otimes_i F$ ,  $\mathbf{1}^{od}$  for the identity operator on  $Z^{od}$  and  $\mathbf{i}^{od}$  for  $\mathbf{1}^{od} \otimes_i \tilde{i} : X^{od} \rightarrow Y^{od}$ . Similarly to what was said about  $X$  and  $Y$ ,  $X^{od}$  and  $Y^{od}$  are contractive  $c_0$ -modules with respect to the same outer multiplication as for  $X$  and  $Y$  (cf. above), and  $\mathbf{i}^{od}$  is an isometric morphism of  $c_0$ -modules. Besides, we introduce the operator

$\mathbf{i}_{an} : (X^{od})_{an} \rightarrow (Y^{od})_{an}$ , which is well defined by taking a coset  $x + (X^{od})_{es}$  to  $\mathbf{i}^{od}(x) + (Y^{od})_{es}$ .

Our next claim is

2<sup>0</sup>. The operator  $\mathbf{i}_{an} \otimes_p \mathbf{1}_\bullet : (X^{od})_{an} \otimes_p (Z^{ev})_{an} \rightarrow (Y^{od})_{an} \otimes_p (Z^{ev})_{an}$  is not an isometry.

Denote the sequence  $(1, 0, 1, 0, 1, \dots) \in Z^{od}$  by  $\tilde{1}^{od}$ . Consider the operator  $\mathbf{s}_E : E \rightarrow (X^{od})_{an}$ , taking a vector  $x$  to the coset  $(\tilde{1}^{od} \otimes_i x) + (X^{od})_{es}$ , and then  $\mathbf{s}_E \otimes_p \mathbf{1}_\bullet : E \otimes_p (Z^{ev})_{an} \rightarrow (X^{od})_{an} \otimes_p (Z^{ev})_{an}$ . At first we shall show, as an intermediate step, that the latter operator is an isometry.

For this aim, using the Hahn-Banach Theorem, introduce the functional  $h : Z^{od} \rightarrow \mathbb{C}$  of norm 1, which takes the subspace  $(Z^{od})_{es} = c_0 \cap Z^{od}$  to 0 and takes  $\tilde{1}^{od}$  to 1. It gives rise to the operator  $\mathbf{t}_E^0 := h \otimes_i \mathbf{1}_E : Z^{od} \otimes_i E \rightarrow \mathbb{C} \otimes_i E$ , that is  $\mathbf{t}_E^0 : X^{od} \rightarrow E$ . The latter evidently takes  $(X^{od})_{es}$  to 0 and thererfore generates the operator  $\mathbf{t}_E := (X^{od})_{an} \rightarrow E$ , well defined by taking the coset  $u + (X^{od})_{es}; u \in X^{od}$  to  $\mathbf{t}_E^0(u)$ . Since  $\mathbf{s}_E$  and  $\mathbf{t}_E$  are, of course, contractive, the same is true with  $\mathbf{s}_E \otimes_p \mathbf{1}_\bullet$  and  $\mathbf{t}_E \otimes_p \mathbf{1}_\bullet$ . But the composition  $(\mathbf{t}_E \otimes_p \mathbf{1}_{an})(\mathbf{s}_E \otimes_p \mathbf{1}_{an})$  is the identity operator on  $E \otimes_p (Z^{ev})_{an}$ . This implies that the former of these two is an isometry (and the latter is a coisometry).

In a similar way, we introduce the operator  $\mathbf{s}_F \otimes_p \mathbf{1}_\bullet : F \otimes_p (Z^{ev})_{an} \rightarrow (Y^{od})_{an} \otimes_p (Z^{ev})_{an}$  and show that it is also an isometry. Consequently, in the diagram

$$\begin{array}{ccc} E \otimes_p (Z^{ev})_{an} & \xrightarrow{\mathbf{s}_E \otimes_p \mathbf{1}_\bullet} & (X^{od})_{an} \otimes_p (Z^{ev})_{an} \\ \tilde{i} \otimes_p \mathbf{1}_\bullet \downarrow & & \downarrow i_{an} \otimes_p \mathbf{1}_\bullet \\ F \otimes_p (Z^{ev})_{an} & \xrightarrow{\mathbf{s}_F \otimes_p \mathbf{1}_\bullet} & (Y^{od})_{an} \otimes_p (Z^{ev})_{an} \end{array}$$

the horizontal arrows depict isometries. Further, our diagram is obviously commutative. Thus it shows that the operator, depicted by the left vertical arrow, is isometrically equivalent to a birestriction of the operator, depicted by the right vertical arrow. But we already know that the former one is an isometry. Therefore the same is true for the latter.

We turn to the next claim.

3<sup>0</sup>. The morphism  $\mathbf{i}^{od} \otimes_{c_0} \mathbf{1}^{ev} : X^{od} \otimes_{c_0} Z^{ev} \rightarrow Y^{od} \otimes_{c_0} Z^{ev}$  is not isometric.

The set of odd natural numbers is the support of both  $X^{od}$  and  $Y^{od}$  whereas the set of even natural numbers is the support of  $Z^{ev}$ . Therefore Proposition 4.2 provides the isometric isomorphisms  $I_{X^{od}, Z^{ev}} : X^{od} \otimes_{c_0} Z^{ev} \rightarrow (X^{od})_{an} \otimes_p (Z^{ev})_{an}$  and  $I_{Y^{od}, Z^{ev}} : Y^{od} \otimes_{c_0} Z^{ev} \rightarrow (Y^{od})_{an} \otimes_p (Z^{ev})_{an}$ , well defined as it was indicated. Looking at the respective commutative diagram, we see that these isomorphisms implement

the isometric equivalence between the operators  $\mathbf{i}_{c_0}^{od} \otimes \mathbf{1}^{ev}$  and  $\mathbf{i}_{an} \otimes \mathbf{1}_p$ . Thus the present claim follows from the previous one.

4<sup>0</sup>. The end of the proof.

Let  $\rho^{od} : Z^{od} \rightarrow Z$  and  $\rho^{ev} : Z^{ev} \rightarrow Z$  be the natural embeddings. Set  $\rho_X^{od} := \rho^{od} \otimes_i \mathbf{1}_E$ ,  $\rho_Y^{od} := \rho^{od} \otimes_i \mathbf{1}_F$ ; these maps are obviously morphisms of  $c_0$ -modules. Consider the diagram

$$\begin{array}{ccc} X^{od} \otimes_{c_0} Z^{ev} & \xrightarrow{\rho_X^{od} \otimes \rho^{ev}} & X \otimes_{c_0} Z \\ \mathbf{i}_{c_0}^{od} \otimes \mathbf{1}^{ev} \downarrow & & \downarrow \mathbf{i}_{c_0} \otimes \mathbf{1} \\ Y^{od} \otimes_{c_0} Z^{ev} & \xrightarrow{\rho_Y^{od} \otimes \rho^{ev}} & Y \otimes_{c_0} Z \end{array}$$

Observe that its horizontal arrows depict isometries. Indeed, introduce the operators  $\sigma^{od} : Z \rightarrow Z^{od} : (\xi_1, \xi_2, \xi_3, \dots) \mapsto (\xi_1, 0, \xi_2, 0, \xi_3, \dots)$ ,  $\sigma^{ev} : Z \rightarrow Z^{ev} : (\xi_1, \xi_2, \xi_3, \dots) \mapsto (0, \xi_1, 0, \xi_2, 0, \xi_3, \dots)$  and set  $\sigma_X^{od} := \sigma^{od} \otimes_i \mathbf{1}_E : Z \otimes_i E \rightarrow Z^{od} \otimes_i E$ . Obviously, the operator  $\sigma_X^{od} \otimes_{c_0} \sigma^{ev}$  is contractive, and the same is true with  $\rho_X^{od} \otimes_{c_0} \rho^{ev}$ . But the composition

$$(\sigma_X^{od} \otimes_{c_0} \sigma^{ev})(\rho_X^{od} \otimes_{c_0} \rho^{ev}) = [(\sigma^{od} \rho^{od}) \otimes_i \mathbf{1}_E] \otimes_{c_0} (\sigma^{ev} \rho^{ev})$$

is the identity operator on  $X^{od} \otimes_{c_0} Z^{ev}$ . This implies that the right factor, in our case  $\rho_X^{od} \otimes_{c_0} \rho^{ev}$ , is an isometry (whereas the left factor is a coisometry). Similarly,  $\rho_Y^{od} \otimes_{c_0} \rho^{ev}$  is an isometry as well.

Our diagram is clearly commutative, and, by the previous claim, its left vertical arrow does not depict an isometry. Hence the same is true with its right vertical arrow (cf. the end of the proof of Claim 2). The rest is clear.  $\triangleright$

**Remark.** The extreme flatness is a recent stronger version of a much older and more investigated notion of a strict flatness, that was mentioned in the introduction. We recall that the definition of a strictly flat module resembles that of an extremely flat module; one must only replace the word ‘isometric’ by ‘topologically injective’ (see., e.g., [4]).

The module  $l_\infty$ , as every normed module over the amenable algebra  $c_0$ , is (just) flat in the standard sense of [3] [4][17]. At the same time, by Theorem 4.3, it is not extremely flat. Here we want to note that one can show, using practically the same argument, as in the proof of the latter theorem, that it is not strictly flat as well. The only difference is that in the very beginning one must use a somewhat stronger property of  $Z := l_\infty/c_0$  than was employed before. Namely, there exist

normed spaces  $E, F$  and topologically injective operator  $\mathbf{i} : E \rightarrow F$  such that  $\mathbf{i} \otimes_p \mathbf{1}_Z$  is not topologically injective. This is because  $l_\infty/c_0$ , being, in the terminology of [15], an  $\mathcal{L}_\infty^g$ -space, can not be an  $\mathcal{L}_1^g$ -space (see Cor. 23.3(4) *idem*) This means, by Cor. 23.5(1) *idem*, that the operation  $\otimes_p l_\infty/c_0$  ‘does not respect subspaces isomorphically’ or, in our terminology,  $l_\infty/c_0$  is not a strictly flat normed space ( $\mathbb{C}$ -module). The subsequent constructions and “claims” are, up to obvious modifications, the same.

## References

- [1] A. Ya. Helemskii. *Quantum Functional Analysis: non-coordinate approach*. AMS, Providence, R.I., 2010.
- [2] M. A. Rieffel. Induced Banach representations of Banach algebras and locally compact groups, J. Funct. Anal., 1 (1967) 443-491.
- [3] A. Ya. Helemskii. *The Homology of Banach and Topological Algebras*. Kluwer, Dordrecht, 1989.
- [4] A. Ya. Helemskii. *Banach and locally convex algebras*. Clarendon Press. Oxford. 1993.
- [5] A. Grothendieck. Une caractérisation vectorielle-métrique des espaces  $L^1$ , Canadian J. Math., 7 (1955) 552-561.
- [6] M. M. Day. *Normed Linear Spaces*. Springer-Verlag. Berlin. 1973. (3rd ed.)
- [7] Z. Semadeni. *Banach Spaces of Continuous Functions*. PWN. Warszawa. 1971.
- [8] J. Cigler, V. Losert, P. Michor. *Banach modules and functors on categories of Banach spaces*. Marcel Dekker, New York, 1979.
- [9] A. Ya. Helemskii. *Lectures and exercises on functional analysis*. AMS, Providence, R.I., 2005.
- [10] M. A. Rieffel. Multipliers and tensor products of  $L^p$ -spaces of locally compact groups, Studia Math., 33 (1969) 71-82.
- [11] A. Ya. Helemskii. Extreme flatness of normed modules and Arveson-Wittstock type theorems, arXiv math/060208v1,5. Feb. 2006. J. Operator Theory, to appear.
- [12] E. G. Effros, Z.-J. Ruan. *Operator spaces*. Clarendon Press. Oxford. 2000.

- [13] G. Wittstock. Injectivity of the module tensor product of semi-Ruan modules, J. Operator Theory, to appear.
- [14] M. Takesaki. *Theory of operator algebras I*. Springer-Verlag, Berlin, 1979.
- [15] A. Defant, K. Floret. *Tensor Norms and Operator Ideals*. North-Holland. Amsterdam. 1993.
- [16] R. A. Ryan. *Introduction to Tensor Products of Banach Spaces*. Springer-Verlag, Berlin, 2002.
- [17] V. Runde. *Lectures on Amenability*. Springer-Verlag, Berlin, 2002.